

MAS 117**Introduction to Algebra****Solutions 5****February 2007**

1 By long division,

$$\begin{aligned}3335 &= 1 \times 2262 + 1073 \\2262 &= 2 \times 1073 + 116 \\1073 &= 9 \times 116 + 29 \\116 &= 4 \times 29\end{aligned}$$

so $\gcd(3335, 2262) = 29$.

Then

$$\begin{aligned}29 &= 1073 - 9 \times 116 \\&= 1073 - 9 \times (2262 - 2 \times 1073) = 19 \times 1073 - 9 \times 2262 \\&= 19 \times (3335 - 2262) - 9 \times 2262 = 19 \times 3335 - 28 \times 2262\end{aligned}$$

So $u = -28$, $v = 19$.

2 By long division,

$$\begin{aligned}x^4 + 4x^3 + 3x^2 - 2x - 1 &= (x+3)(x^3 + x^2 - 5x - 2) + (5x^2 + 15x + 5), \\x^3 + x^2 - 5x - 2 &= (x-2)(x^2 + 3x + 1),\end{aligned}$$

so $\gcd(x^4 + 4x^3 + 3x^2 - 2x - 1, x^3 + x^2 - 5x - 2) = x^2 + 3x + 1$. [Note that we can divide through by the factor 5 in going from the first line to the second to avoid having to deal with fractions. If we didn't divide by 5, it would not be correct to say that $5x^2 + 15x + 5$ is the gcd, since part of our definition of the gcd is that it should be monic!]

Then

$$x^2 + 3x + 1 = \frac{1}{5}(x^4 + 4x^3 + 3x^2 - 2x - 1) - \frac{1}{5}(x+3)(x^3 + x^2 - 5x - 2),$$

so $u(x) = -\frac{1}{5}(x+3)$, $v(x) = \frac{1}{5}$.

3 (a) Let $A(n)$ be the statement “the value of $n!$ is defined by the given rule”, that is, we can calculate the value from the given information.

- *Starting the induction:* $A(0)$ is true, since the rule gives us the value of $0!$, namely $0! = 1$.
- *The inductive step:* Suppose that $A(n)$ is true, that is, the value of $n!$ is defined. Then $(n+1)! = (n+1) \times n!$, so the value of $(n+1)!$ is also defined; that is, $A(n+1)$ is true.

By induction, $A(n)$ is true for all $n \geq 0$.

(b) Let $A(n)$ be the statement $2^{n-1} \leq n! \leq n^n$.

- *Starting the induction:* We are told to prove the statement for $n \geq 1$, so we start the induction for $n = 1$. In this case, we have

$$2^0 = 1 \quad 1! = 1 \times 0! = 1 \quad 1^1 = 1,$$

so $A(1)$ asserts that $1 \leq 1 \leq 1$, which is true.

- *The inductive step:* Suppose that $n \geq 2$, and that $A(n)$ holds, i.e. that $2^{n-1} \leq n! \leq n^n$.

– We have $2 \leq (n+1)$, so

$$2^n = 2 \times 2^{n-1} \leq (n+1) \times n! = (n+1)!.$$

– We have $n \leq n+1$, so

$$(n+1)! = (n+1) \times n! \leq (n+1) \times n^n \leq (n+1) \times (n+1)^n = (n+1)^{n+1}.$$

So $A(n+1)$ is true.

By induction, $A(n)$ is true for all n .

4 Think of an operation on a set as a black box, where we put in two inputs x and y and get just one output $x \circ y$. This particular black box is a rather stupid one, since whatever inputs we put in, the output is always p .

Associative law? Yes, since $x \circ (y \circ z) = p = (x \circ y) \circ z$ for all x, y, z .

Commutative law? Yes, since $x \circ y = p = y \circ x$ for all x, y .

Identity law? There is no identity. For an identity element e would have to satisfy $e \circ q = q$, but neither $e = p$ nor $e = q$ works here.

Inverse law? Not applicable since there is no identity.